Lagrangian approach to integrable systems yields new symplectic structures for KdV

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In the literature on integrable systems we find Hamiltonian operators without explanation. There is a notable silence on Lagrangians from which these Hamiltonian and symplectic structures can be derived. We show that starting with Lagrangians, which turn out to be degenerate, the Hamiltonian operators for integrable systems can be constructed using Dirac's theory of constraints. We illustrate this by giving a systematic discussion of the first Hamiltonian structure of KdV. First by Dirac's theory and then applying the covariant Witten-Zuckerman theory of symplectic structure we arrive at its flux. Then we turn to a new Lagrangian for KdV recently obtained by Pavlov and derive the corresponding new symplectic structure for KdV. We show that KdV admits infinitely many Lagrangian formulations and therefore infinitely many symplectic structures.

1 Introduction

While there is no precise definition of complete integrability there are many properties that we expect from a completely integrable system which in fact enable us to recognize it as such. Bi-Hamiltonian structure for which we have the celebrated theorem of Magri [1] is a case in point. Non-linear evolution equations that can be cast into Hamiltonian form in two inequivalent but compatible ways admit a full set of conserved quantities required by complete integrability. These conserved Hamiltonians are in involution with respect to Poisson brackets defined by both Hamiltonian structures.

We recall from earliest school that Hamiltonian structure is derivable from a Lagrangian, yet in complete integrability there is a notable, even deafening silence on Lagrangians. Why?

It turns out that in the variational formulation of integrable equations the Lagrangians are degenerate and therefore require the use of Dirac's theory of constraints [2] in order to achieve Hamiltonian form. On the other hand, degenerate Lagrangians for integrable systems always admit a complete set of second class constraints which enables us to eliminate all the momenta and write the Hamiltonian equations of motion solely in terms of the original variables. In effect this means that we can often guess the Dirac bracket without recourse to Dirac's theory. Thus in the literature on integrable systems Dirac's theory of constraints hardly even gets a nod while Dirac brackets are generally known as Hamiltonian operators. Even in the excellent book of Dorfman [3] that she was able to complete before her untimely death, Dirac's name appears in the title but in its pages no reference can be found to a single degenerate Lagrangian for which the Hamiltonian structure is obtained through Dirac's theory of constraints. So, for both reasons, namely the fact that Dirac's theory is not in the arsenal of most mathematicians as well as the convenience afforded by second class constraints of eliminating the momenta completely, the present literature contains very little on degenerate Lagrangians for integrable systems. It was, however, my original motivation for going into integrable systems [4].

Inevitably something as fundamental as the Lagrangian would make a spectacular come-back and indeed it has. The covariant approach to symplectic structure which has relatively recently been formulated by Witten [5] and Zuckerman [6] employs the Lagrangian as its head-piece.

The usual Hamiltonian approach is not covariant because it singles out an independent variable as "time" with respect to which the evolutionary system will be defined. Dirac was well aware of this short-coming because he was ultimately interested in the Hamiltonian formulation of Einstein's general relativity [7] and in his book [2] complains several times about the fact that the Hamiltonian formalism is not covariant. The Witten-Zuckerman covariant theory of symplectic structure is very much in the spirit of Dirac's work. It starts with a Lagrangian and in the construction of the symplectic 2-form the most crucial role is played by the boundary terms in the first variation of the action. We shall now illustrate the prominence of the Lagrangian in the covariant theory of symplectic structure by applying the Witten-Zuckerman construction of the symplectic 2-form to the grandmother of all integrable systems, namely KdV. We shall show that not only do we recover some familiar results but we shall also able to present some new symplectic structures starting from new Lagrangians for the KdV equation. Shockingly enough, we shall find that there are still unknown fundamental results in the theory of KdV.

2 KdV as bi-Hamiltonian system

The bi-Hamiltonian structure of KdV was the first one to be discovered and has consequently served as the model for the multi-Hamiltonian structure of all integrable systems. It is well known that the KdV equation

$$u_t + 6\, u\, u_x + u_{xxx} = 0 \tag{1}$$

can be cast into the form of Hamilton's equations

$$u_{\,t} = \mathbf{X}(u) = \{u, H\}_D = J\ \delta_u H \tag{2}$$

where $\mathbf{X}$ is the vector field defining the flow which is manifest from eq.(1), J is the Hamiltonian operator defining the Poisson bracket and δ_u denotes $\delta/\delta u$, the variational derivative with respect to u. The Hamiltonian operator which is a skew-symmetric matrix of differential operators satisfying the Jacobi identities is simply obtained from the Dirac bracket as the subscript D in eq.(2) indicates. However, this was not the historical route to the Hamiltonian formulation of KdV which started with an important paper by Gardner [10] and independently by Zakharov and Fadeev [11] where they showed that the infinite set of conserved Hamiltonians $H_i, i = 1, \ldots, \infty$ commute with respect to Poisson brackets defined by

$$\left\{ H_i, \frac{d}{dx}\, H_k \right\}_P = 0, \tag{3}$$

which, as we shall soon show explicitly, is of course a particular example of the Dirac bracket. Thus

$$J_0 = \frac{d}{dx} \tag{4}$$

is the first Hamiltonian operator for KdV. The next important development was an unpublished but widely known result of Lenard, namely the recursion operator for KdV. Magri [1] realized that it leads to the second Hamiltonian operator for KdV

$$J_1 = \frac{d^3}{dx^3} + 2u\,\frac{d}{dx} + \frac{d}{dx}\,2u \tag{5}$$

and was able to formulate his theorem on bi-Hamiltonian structure. Lenard's recursion operator is simply

$$\mathcal{R} = J_1\;(J_0)^{-1} \tag{6}$$

and in the bi-Hamiltonian formulation of KdV the recursion relation for conserved Hamiltonians is given by

$$u_{\,t_n} = J_0\;\delta_u H_{n+1} = J_1\;\delta_u H_n \tag{7}$$

with Hamiltonian densities given by

$$\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2}u^2, && (8)\\
\mathcal{H}_1 &= u^3 - \frac{1}{2}u_x^{\,2} && (9)\\
\mathcal{H}_2 &= \frac{5}{2}\,u^4 - 5\,u\,u_x^{\,2} + \frac{1}{2}\,u_{xx}^{\;\;2} && (10)\\
.. &\quad\; ...
\end{aligned}$$

which determines the higher flows in the KdV hierarchy.

In effect, there was no derivation of these Hamiltonian operators. When we are presented with a result like this all we can do is to check the properties of skew-symmetry and Jacobi identities required of a Poisson bracket. Then we must keep quiet, but it is an uneasy quiet.

3 Dirac bracket is the Hamiltonian operator

The first correct identification of the Hamiltonian operator (4) as the Dirac bracket appeared in a paper by Macfarlane [8] which, as far as I know, is unpublished and has not been given the attention that it deserves. The first systematic application of Dirac's theory of constraints to the degenerate Lagrangian for KdV was given in [9].

We start with the variational formulation of KdV. For this purpose we need to introduce

$$u = \phi_x , \tag{11}$$

the Clebsch velocity potential. The potential KdV is

$$\phi_t + 3\,\phi_x^{\,2} + \phi_{xxx} = 0 \tag{12}$$

and it can be directly verified that the equations of motion following from the variational principle

$$\delta I = 0\,, \qquad I = \int \mathcal{L}\; dt\, dx \tag{13}$$

with Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2}\,\phi_t\,\phi_x + \phi_x^3 - \frac{1}{2}\,\phi_{xx}^{\,2} \tag{14}$$

yield KdV. This Lagrangian is degenerate because its Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial \phi_t^{\,2}} = 0 \tag{15}$$

vanishes identically. Alternatively, the canonical momentum given by

$$\pi = \frac{\partial \mathcal{L}_0}{\partial \phi_t} = \frac{1}{2}\,\phi_x \tag{16}$$

cannot be inverted for the velocity ϕ_t and we have a degenerate Lagrangian system. Therefore the Hamiltonian formulation of the Lagrangian (14) requires the use of Dirac's theory of constraints.

Following Dirac [2] we introduce the primary constraint that results from the definition of the momentum (16)

$$\Phi = \pi - \frac{1}{2}\,\phi_x \tag{17}$$

and *a priori* require that it vanish either "weakly," or as a "strong" equation depending on whether or not the constraint is first, or second class in Dirac's terminology. This is determined by evaluating the Poisson bracket of the constraints. If the result should vanish modulo the constraints then it is a first, otherwise second class constraint. A second class constraint is an equation that can be solved to eliminate a canonical variable.

In order to calculate the Poisson bracket of the constraints we use the canonical Poisson bracket relations

$$\{\pi(x), \phi(y)\}_P = \delta(x-y) \tag{18}$$

with all others vanishing. The result

$$\{\Phi(x), \Phi(y)\}_P = \frac{1}{2}\delta_y(x-y) - \frac{1}{2}\delta_x(y-x) \tag{19}$$

shows that the constraint (17) is second class as it does not vanish by virtue of the constraint itself. A word of caution, that I owe to my friend Galvão [12], is in order here: One should not simplify the Poisson brackets of the constraints (19) using the rules for manipulating distributions at intermediate stages of calculation. In our case we shall need to invert this distribution and therefore we must keep the cumbersome appearance of eq.(19). Simplifications should be made only at final results.

The total Hamiltonian of Dirac is given by

$$\begin{aligned} H_T &= \int (\pi\,\phi_t - \mathcal{L} + \lambda\,\Phi)\,dx && (20)\\ &= \int \left[-\phi_x^3 + \frac{1}{2}\,\phi_{xx}{}^2 + \lambda\left(\pi - \frac{1}{2}\phi_x\right)\right] dx && (21) \end{aligned}$$

where λ is a Lagrange multiplier. The condition that the constraint is maintained in time

$$\{\Phi(x), H_T\} = 0 \tag{22}$$

gives rise to no further constraints which would have been secondary constraints. Instead, using eqs.(19), we find that the Lagrange multiplier is completely determined from eq.(22)

$$\lambda = -3\,\phi_x^2 - \phi_{xxx} \tag{23}$$

which is expected, since the constraint and therefore the total Hamiltonian is linear in the momenta the correct equation of motion (12) will result only if the Lagrange multiplier is simply the component of the vector field defining the flow for potential KdV. The total Hamiltonian density of Dirac

$$\mathcal{H}_T = \frac{1}{2}\,\phi_x^3 - \pi\left(3\,\phi_x^2 + \phi_{xxx}\right) \tag{24}$$

follows from the substitution of the Lagrange multiplier (23) in eq.(21). Now the check that all the Hamiltonian equations of motion are satisfied with the Hamiltonian (24) is straight-forward. We can summarize all of them in Hamilton's equations

$$\mathcal{A}_t = \{\mathcal{A}, H_T\} \tag{25}$$

where $\mathcal{A}$ is any smooth functional of the canonical variables ϕ, π and their derivatives.

There is, however, one further and very important simplification that we can carry out because in Dirac's theory second class constraints hold as strong equations. This fact enables us to eliminate the momentum in the total Hamiltonian (24) using the solution of eq.(16). Thus we find

$$\mathcal{H}_T = -\,\phi_x^{\,3} + \frac{1}{2}\,\phi_{xx}^{\;\;2} \tag{26}$$

for Dirac's total Hamiltonian density. Apart from an overall minus sign, this is just the Hamiltonian function (9) for the first Hamiltonian structure of KdV.

The Dirac bracket is the projection of the Poisson bracket from phase space onto the hyper-surface defined by the constraint. Given any two differentiable functionals of the canonical variables $\mathcal{A}$ and $\mathcal{B}$, the Dirac bracket is defined by

$$\begin{aligned} \{\mathcal{A}(x), \mathcal{B}(y)\}_D = \;& \{\mathcal{A}(x), \mathcal{B}(y)\} \\ & - \int \{\mathcal{A}(x), \Phi(z)\} J(z,w) \{\Phi(w), \mathcal{B}(y)\}\, dz\, dw \end{aligned} \tag{27}$$

where J is the inverse of the matrix of Poisson brackets of the constraints. The definition of the inverse is simply

$$\int \{\Phi(x), \Phi(z)\} J(z,y)\; dz = \delta(x-y) \tag{28}$$

which results in a differential equation to be solved for J. Starting with the Poisson bracket relation (19) we find that eq.(28) can be solved readily to yield

$$J(x,y) \;\equiv\; J(x)\delta(x-y) \tag{29}$$

$$\;=\; \left(\frac{d}{dx}\right)^{-1} \delta(x-y) = \theta(x-y) \tag{30}$$

where θ is the Heaviside unit step function and $(d/dx)^{-1}$ is the principal value integral [13]. With the definition of the Hamiltonian operator $J(x)$ given above this principal value integral is the first Hamiltonian operator of KdV in terms of the Clebsch potential.

Now we need to make contact with the first Hamiltonian operator (4) for KdV which is expressed in terms of the velocity field u rather than its potential ϕ. In view of the definition of the potential (11) and the transformation properties of Hamiltonian operators, we find

$$J_{u(x)} = \frac{d}{dx} \left(\frac{d}{dx}\right)^{-1} \frac{d}{dx} = \frac{d}{dx} \tag{31}$$

which is the same as (4). This is the derivation of the first Hamiltonian operator for KdV from first principles, *i.e.* from a Lagrangian approach. I have gone into it in some detail not only because it is a nice illustrative example but also because carrying out the same construction for the second Hamiltonian operator (5) is a highly non-trivial unsolved problem.

4 Symplectic form of KdV

The Hamiltonian operator maps differentials of functions into vector fields and in the opposite direction we have the symplectic 2-form ω which is the principal geometrical object in the theory of symplectic structure. The symplectic 2-form is closed

$$\delta\omega = 0\,, \tag{32}$$

which is equivalent to the Jacobi identities satisfied by the Dirac bracket. By Poincaré's lemma, in a local neighborhood, ω can be written as

$$\omega = \delta\alpha, \tag{33}$$

where α is a 1-form. The statement of the symplectic structure of the equations of motion consists of

$$i_{\mathbf{X}}\omega = \delta H \tag{34}$$

which is obtained by the contraction of the symplectic 2-form ω with the vector field $\mathbf{X}$ defining the flow.

The symplectic 2-form is given by

$$\omega = \frac{1}{2} \int \delta\phi(x) \wedge K(x,y)\, \delta\phi(y)\; dy \tag{35}$$

where $K(x,y)$ is the inverse of $J(x,y)$. This route of arriving at the symplectic 2-form through the Hamiltonian operator is commonplace in the literature of integrable systems. But in the Lagrangian approach it is absurd because we have the answer already in eq.(28): Hamiltonian operators are Dirac brackets for systems subject to second class primary constraints which are obtained by *inverting* the Poisson bracket of the constraints! Hence

$$K(x,y) \equiv \{\Phi(x), \Phi(y)\}, \tag{36}$$

the symplectic 2-form density can be obtained directly from the Poisson bracket of second class constraints. In the case of the first Hamiltonian structure of KdV from eq.(19) we find

$$\omega_0 = \delta\phi \wedge \delta\phi_x \tag{37}$$

for the first symplectic 2-form of KdV.

It remains to check Hamilton's equations in the symplectic form (34) for KdV using this symplectic 2-form and H_1. For this purpose we need to contract the 2-form (37) with the vector field

$$\mathbf{X} = -\left(3\,\phi_x^2 + \phi_{xxx}\right) \frac{\delta}{\delta\phi} \tag{38}$$

defining the flow for potential KdV. Recalling

$$i_{\delta/\delta\phi(y)} \delta\phi_x = \delta_x(x-y), \tag{39}$$

it follows immediately that eq.(34) is satisfied with H_1 given by (9).

5 Witten-Zuckerman 2-form

Time plays a privileged role in Hamiltonian mechanics. While this presents no problem for systems with finitely many degrees of freedom, in field theory it has the severe disadvantage of non-covariance. In order to remedy this situation Witten [5] and Zuckerman [6] have introduced the conserved current 2-form which provides an elegant covariant formulation of symplectic structure. The Witten-Zuckerman 2-form ω^μ, where in our case μ ranges over two values t and x, satisfies

$$\begin{aligned} \delta\omega^\mu &= 0, && (40)\\ \omega^\mu_{\ ,\mu} &= 0, && (41) \end{aligned}$$

the properties of closure and conservation. Hamilton's equations can now be written in the covariant form

$$i_{\mathbf{X}}\,\omega^\mu = \delta H^\mu \tag{42}$$

where H^t is the familiar Hamiltonian function and H^x is its flux.

The Witten-Zuckerman current 2-form ω^μ is derived from the variational principle underlying the equations of motion. In the case of the Lagrangian (14) we need to consider its first variation assuming the equation of motion (12) and its Jacobi equation

$$\delta\phi_t + 6\,\phi_x\,\delta\phi_x + \delta\phi_{xxx} = 0. \tag{43}$$

Then the first variation of the Lagrangian reduces to a conservation law

$$\delta\mathcal{L} = \alpha^t_{\ ,t} + \alpha^x_{\ ,x} \tag{44}$$

where α^μ is a 1-form. For the Lagrangian (14) we find

$$\begin{aligned} \alpha^t &= \frac{\partial\mathcal{L}}{\partial\phi_t}\,\delta\phi \;=\; \tfrac{1}{2}\,\phi_x\,\delta\phi,\\ \alpha^x &= \frac{\partial\mathcal{L}}{\partial\phi_x}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{xx}}\delta\phi_x - \left(\frac{\partial\mathcal{L}}{\partial\phi_{xx}}\right)_x\delta\phi + \ldots \qquad (45)\\ &= \left(\tfrac{1}{2}\,\phi_t + 3\,\phi_x^2 + \phi_{xxx}\right)\delta\phi - \phi_{xx}\,\delta\phi_x. \end{aligned}$$

and the symplectic current 2-form ω^μ is given by eq.(33). We find that

$$\begin{aligned} \omega^t &= \frac{1}{2}\,\delta\phi_x\wedge\delta\phi && (46)\\ \omega^x &= 3\,\phi_x\,\delta\phi_x\wedge\delta\phi + \delta\phi_x\wedge\delta\phi_{xx} + \frac{1}{2}\,\delta\phi_{xxx}\wedge\delta\phi && (47) \end{aligned}$$

where we have used the Jacobi equation (43). Note that the freedom of adding an arbitrary divergence term to the Lagrangian disappears in the Witten-Zuckerman symplectic 2-form. We can check that eq.(47) remains invariant under the replacement of $-\frac{1}{2}\phi_{xx}^{\ 2}$ in the Lagrangian (14) by its divergence equivalent $\frac{1}{2}\phi_x\phi_{xxx}$.

It is obvious that both components of ω^μ satisfy the closure property (40) but the check of conservation law (41) requires use of the Jacobi equation (43) again. Finally, we need to check Hamilton's equations in the covariant symplectic form of eqs.(42). We have already checked the t component in section 4. Using (39) and its generalization to higher derivatives, we find that the contraction of (47) with (38) yields δH_1^x where

$$H_1^x = \frac{9}{2}\,\phi_x^{\ 4} + 3\,\phi_x^{\ 2}\phi_{xxx} - 6\,\phi_x\,\phi_{xx}^{\ 2} - \phi_{xx}\phi_{4x} + \frac{1}{2}\,\phi_{xxx}^{\ 2} \tag{48}$$

is the familiar flux of H_1.

We conclude by remarking again that the time component of the Witten-Zuckerman 2-form (46) is precisely the symplectic 2-form (37) obtained from Dirac's theory of constraints.

6 Pavlov's new KdV Lagrangian

We have derived the first Hamiltonian operator for KdV as the Dirac bracket for the degenerate Lagrangian (14) and showed that it can be obtained from the covariant Witten-Zuckerman theory as well. But now we know that there is also a second Hamiltonian operator for KdV. So the question naturally arises as to "What is the Lagrangian for which Magri's Hamiltonian operator (5) is the Dirac bracket?"

I have asked this question, thinking that it is a rhetorical question, in several talks that I gave on integrable systems. Last time I did so, at NEEDS 2000 in Gökova, my friend Pavlov [14] got up to say that there are indeed many Lagrangians for KdV and showed me that

$$\mathcal{L}_{-1} = \left(\phi_x^{\ 2} + \frac{1}{2}\,\phi_{xxx}\right)\phi_t + \left(\frac{5}{2}\,\phi_x^{\ 4} - 5\,\phi_x\,\phi_{xx}^{\ 2} + \frac{1}{2}\,\phi_x^3\right) \tag{49}$$

is another Lagrangian for KdV. Indeed one can verify it directly. But the point is that this Lagrangian is not just an inspired guess. There is a direct derivation of (49) which opens up the flood gates to an infinite family of Lagrangians for KdV as the subscript -1 indicates.[1]

The derivation of Pavlov's new Lagrangian for KdV is based the Lenard recursion relation (7). To understand this, let us first go back to the classical Lagrangian (14) and see how, in retrospect, we would derive it. For $n = 1$ the recursion relation (7) can be written in the form

$$u_t = (\phi_x)_t = J_0\,\delta_u H_1 = \delta_\phi\, H_1(u = \phi_x) \tag{50}$$

in view of eqs.(4) and (11). Indeed, in the Lagrangian (14) terms that do not depend on the velocity precisely make up H_1. In order to obtain the full Lagrangian (14) all that remains to be done is to rewrite the left hand side of (50), namely ϕ_{tx} in variational form. In complete analogy, for Pavlov's Lagrangian terms which do not involve the velocity consist of H_2 given by (10), the next conserved Hamiltonian function in the KdV hierarchy. Then we go back to the Lenard recursion relation and write it for $n = 2$

$$\begin{array}{rclcl} & J_0\,\delta_u H_2 & = \quad J_1\,\delta_u H_1 & = & \mathcal{R}\,J_0\,\delta_u H_1 \\ = & \delta_\phi H_2 & & = & \mathcal{R}\,\delta_\phi H_1 = \mathcal{R}\,\phi_{tx} \end{array} \tag{51}$$

and therefore for Pavlov's Lagrangian the factor in front of ϕ_t is simply obtained by writing the action of the recursion operator on ϕ_{tx} in variational form. From this construction it is manifest that starting with H_n all we need to do in order to obtain a new Lagrangian $\mathcal{L}_{-n}$ for KdV is to write $\mathcal{R}^n\,\phi_{tx}$ in variational form. Hence we have

Theorem: *For every Hamiltonian function in the KdV hierarchy, there exists a degenerate Lagrangian that yields KdV as its Euler-Lagrange equation.*

The number of Lagrangians for KdV is therefore infinite in number. However, here we must note that while KdV will be an extremum for the first variation of all these Lagrangians, the Euler equations require something weaker. Namely, the n^{th} power of the action of the recursion operator on KdV should vanish.

[1] I have since received email from Z. Popowicz who has written a computer program to generate Lagrangian after Lagrangian for KdV. Because he is a computer expert he couldn't resist the temptation to invent his own notation. Unfortunately, I found his notation absolutely impossible to decipher. Fortunately, there is no need to do so. In the next paragraph we shall present the derivation of all these Lagrangians.

Finally, we note that the Lenard recursion operator (6) is non-local and its repeated application on ϕ_{tx} will require the introduction of non-local terms in higher Lagrangians. This is not a problem because the original KdV Lagrangian is itself non-local! The Lagrangian (14) is not expressible in terms of the original variable u but by its potential (11), or $\phi = (d/dx)^{-1}\, u$. Higher Lagrangians for KdV can be written in local form by introducing potentials for the Clebsch potential according to $\phi = \psi_x$ etc.

7 New symplectic structure of KdV

Every new Lagrangian for KdV will give rise to a new symplectic structure. This can be obtained either through Dirac's theory of constraints, or directly through the Witten-Zuckerman theory. We shall now briefly discuss both of these procedures for Pavlov's Lagrangian (49). The result will be a new symplectic structure for KdV. It is not the one we were questing after: The new symplectic 2-form is not the inverse of Magri's Hamiltonian operator.

7.1 Dirac constraint analysis

Pavlov's Lagrangian (49) is degenerate because it is linear in the velocity ϕ_t. Its Hessian (15) vanishes identically. So we need to apply Dirac's theory of constraints in order to cast it into Hamiltonian form. The constraint obtained from the definition of momentum that follows from the Lagrangian (49) is given by

$$\Phi = \pi - \phi_x^2 - \frac{1}{2}\,\phi_{xxx} \tag{52}$$

and it is second class because

$$\{\Phi(x), \Phi(y)\}_P = \frac{1}{2}\delta_{xxx}(y-x) - \frac{1}{2}\delta_{yyy}(x-y) + 2\,\phi_x\,\delta_x(y-x) - 2\,\phi_y\,\delta_y(x-y) \tag{53}$$

does not vanish modulo (52). The symplectic 2-form, obtained through (35) and (36) is given by

$$\omega_{-1} = \frac{1}{2}\,\delta\phi_{xxx} \wedge \delta\phi + 2\,\phi_x\,\delta\phi_x \wedge \delta\phi \tag{54}$$

which is manifestly a closed 2-form. Now Hamilton's equations (34) are satisfied with H_2 on the right hand side.

In order to obtain the Hamiltonian operator that corresponds to this new symplectic structure for KdV we should invert the Poisson bracket of the constraints (53) through the definition (28). Here we encounter another surprise because the equation satisfied by the two-point function $J_{-1}(x,y)$ that will lead to the new Hamiltonian operator through the definition in eq.(29) is given by

$$\frac{d^3}{dx^3}J_{-1}(x,y) + 4\,\phi_x \frac{d}{dx}J_{-1}(x,y) + 2\,\phi_{xx}\,J_{-1}(x,y) = \delta(x-y) \qquad (55)$$

which is precisely the equation to be solved for the inverse of Magri's second Hamiltonian operator (5). The inverse of Magri's operator is not known. One could write it as an infinite series which, however, does not appear to be tractable. So we are unable to write the minus first Hamiltonian operator explicitly but we have arrived at a curious result. Namely, *the inverse Magri's Hamiltonian operator (5) is also a Hamiltonian operator for KdV.*

7.2 Witten-Zuckerman analysis

The covariant symplectic 2-form for Pavlov's Lagrangian is derived along the same lines as in section 5. Here we shall only record the final result. The time component of the Witten-Zuckerman symplectic 2-form is the same as the expression in eq.(54) obtained from Dirac's theory. Now, however, we also have its flux

$$\begin{aligned}\omega^x_{-1} \;=\; & (12\,\phi_x^{\,2} + 5\,\phi_{xxx})\,\delta\phi_x \wedge \delta\phi - 6\,\phi_{xx}\,\delta\phi_{xx} \wedge \delta\phi + 5\,\phi_x\,\delta\phi_{xxx} \wedge \delta\phi \qquad (56)\\ & -\frac{1}{2}\,\delta\phi_{5x} \wedge \delta\phi - 4\,\phi_x\,\delta\phi_{xx} \wedge \delta\phi_x - \frac{3}{2}\,\delta\phi_{4x} \wedge \delta\phi_x + \frac{1}{2}\delta\phi_{xxx} \wedge \delta\phi_{xx}.\end{aligned}$$

The check that Hamiltonian equations in the covariant form of eqs.(42) are satisfied with this symplectic 2-form and the second Hamiltonian function H_2 (10) is lengthy but straight-forward. For the space component of the Witten-Zuckerman 2-form it is even lengthier to check that these equations are satisfied with (56) and

$$\begin{aligned}H^x_2 \;=\; & 12\,\phi_x^{\,5} + 10\,\phi_x^{\,3}\phi_{xxx} - 45\,\phi_x^{\,2}\,\phi_{xx}^{\,2} + 8\,\phi_x\,\phi_{xxx}^{\,2} + 5\,\phi_{xx}^{\,2}\,\phi_{xxx}\\ & +\phi_{xxx}\phi_{5x} - \frac{1}{2}\,\phi_{4x}^{\,2}, \qquad (57)\end{aligned}$$

the flux of H_2 on the right hand side of (42).

8 Conclusion

The Lagrangian approach to integrable non-linear evolution equations is one where we can derive everything from first principles. We had earlier used it for various equations ranging from dispersive water waves [15] to Monge-Ampère equations [16], [17] in differential geometry. Here we have tried to illustrate this approach using the first Hamiltonian structure of KdV. Starting with the classical Lagrangian for KdV we applied Dirac's theory of constraints to arrive at its first Hamiltonian operator. We showed that the covariant Witten-Zuckerman theory of symplectic structure not only yields the symplectic 2-form obtained from Dirac's theory but also gives us the flux component of the symplectic 2-form. Then we applied these approaches to Pavlov's new Lagrangian and arrived at a new symplectic structure for KdV. We have arrived at the remarkable result that the inverse Magri's Hamiltonian operator is also a Hamiltonian operator for KdV. Pavlov's procedure for constructing new Lagrangians is applicable to all conserved Hamiltonians of which KdV has infinitely many. Therefore there will be infinitely many symplectic structures for KdV.

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